2. Solution Methods to (M)IP

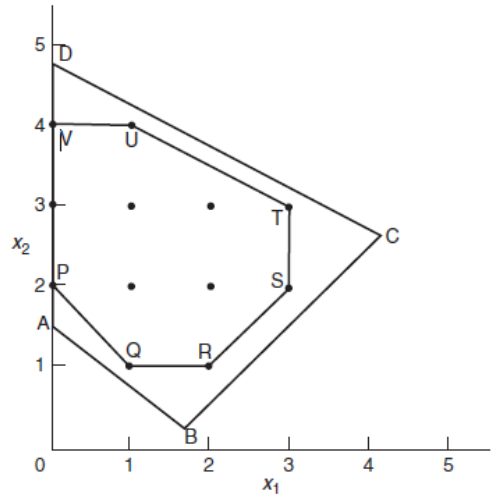
We have formulated several IP/MIP problems to motivate the richness and variety of applications, where our models are mainly addressed by giving insight into the structure of the problem. However, our goal is to solve the problem for an optimal or nearly optimal solution. In contrast to linear programming, most IPs/MIPs can be formulated in several ways. More importantly, in IP/MIP, formulating a **"good"** model is of crucial importance to solving the model.

2.1 Linear Programming Relaxations

Relaxations may weaken either the objective function or the constraints, but the elementary ones we explore in this module nearly all focus on constraints. A constraint relaxation produces an easier model by dropping or easing some constraints.

The best known and most used of all constraint relaxation forms are known as Linear Programming, or more generally, **continuous relaxations**.

Definition 2.1 Continuous Relaxation (LP Relaxation if the given model is an integer linear program) is formed by treating any discrete variables as continuous while retaining all other constraints. The LP relaxation is a less constrained, or more relaxed, version of the IP. (This means that *the feasible region for any IP must be contained in the feasible region for the corresponding LP relaxation.*)



Certainly, [1]no feasible solution is lost by allowing both fractional and integer choices for discrete variables. More important, [2]the relaxed model usually proves significantly more tractable.

Example:
Form the LP relaxation of the following integer program:

$$\begin{aligned} \min \quad & 15x_1 + x_2 - 4x_3 + 10x_4 \\ s.t \quad & x_3 - x_4 \leq 0, \\ & x_1 + 2x_2 + 4x_3 + 8x_4 = 20 \\ & x_1,\ x_2,\ x_3,\ x_4 \in \{0,1\} \end{aligned}$$

Solution:
Following **Definition 2.1**, we replace 0-1 constraints $x_i \in \{0,1\}$ by $x_i \in [0,1]$ to obtain the LP relaxation:

$$\begin{aligned} \min \quad & 15x_1 + x_2 - 4x_3 + 10x_4 \\ s.t \quad & x_3 - x_4 \leq 0, \\ & x_1 + 2x_2 + 4x_3 + 8x_4 = 20 \\ 0 \leq x_1 \leq 1, 0 \leq\ & x_2 \leq 1, 0 \leq\ x_3 \leq 1, 0 \leq\ x_4 \leq 1 \end{aligned}$$

So what does relaxation add to our analysis of discrete optimization models?

Proving Infeasibility

Suppose that a constraint relaxation comes out infeasible, and then it has no solutions at all. Since every solution to the full model must also be feasible in the relaxation, it follows that the original model was also infeasible.

Principle 2.2 If a constraint relaxation is infeasible, so is the full model it relaxes.

Exercise
Prove that the following discrete optimization model is infeasible:

$$\begin{aligned} \min \quad & 8x_1 + 2x_2 \\ s.t \quad & x_1 - x_2 \geq 2 \\ & -x_1 + x_2 \geq 1 \\ & x_1, x_2 \in Z^+ \end{aligned}$$

Bounding Solution

Constraint relaxation expands the feasible set, allowing more candidates for relaxation optimum. The relaxation optimal value must then equal or improve on the best feasible objective value to the true model.

Principle 2.3 The optimal objective value of any relaxation of a maximize model yields an upper bound on the optimal objective value of the full model. The optimal objective value of any relaxation of a minimize model yields a lower bound.

Exercise
Compute (by inspection) the optimal objective value and the LP relaxation bound for the integer program:

$$\begin{aligned} \max \quad & x_1 + x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 \leq 1, \\ & x_1 + x_3 \leq 1, \\ & x_2 + x_3 \leq 1, \\ & x_1, x_2, x_3 \in \{0, 1\} \end{aligned}$$

Optimal Solutions from Relaxations

Sometimes relaxations not only bound the optimal value of the corresponding discrete model but produce an optimal solution.

Principle 2.4 If an optimal solution to a constraint relaxation is also feasible in the model it relaxes, the solution is optimal in that original model.

Exercise Compute (by inspection) the optimal solution to the LP relaxation and determine whether we can conclude that the relaxation optimum is optimal in the original model:

$$\begin{aligned} \max \quad & 20x_1 + 8x_2 + 2x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 \leq 1, \\ & x_1, x_2, x_3 \in \{0, 1\} \end{aligned}$$

Rounded Solutions from Relaxation

When **Principle 2.4** applies, relaxation completely solves a hard discrete optimization model. More commonly, things are not that simple.

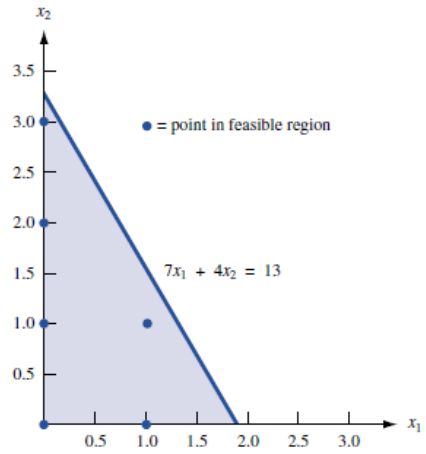
However, as we have the bound of **Principle 2.3**, we may also have a starting point for constructing a good heuristic solution to the full discrete model.

Principle 2.5 Many relaxations produce optimal solutions that are easily" rounded" to **good** feasible solutions for the full model.

Example: (4)

$$\begin{aligned} \max\ z &= 21x_1 + 11x_2 \\ s.t \quad & 7x_1 + 4x_2 \leq 13 \\ & x_1, x_2 \geq 0; x_1, x_2\ integer \end{aligned}$$

From Figure down, we see that the feasible region for this problem consists of the following set of points: S = {(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1)}. Unlike the feasible region for any LP, the one for (4) is not a convex set. By simply computing and comparing the z-values for each of the six points in the feasible region, we find the optimal solution to it is z = 33, x_1 = 0, x_2 = 3. If the feasible region for a pure IP's LP relaxation is bounded, as in (4), then the feasible region for the IP will consist of a finite number of points. Suppose that a naive analyst suggests the following approach for solving an IP: First solve the LP relaxation; then round off (to the nearest integer) each variable that is required to be an integer and that assumes a fractional value in the optimal solution to the LP relaxation. Applying this approach to (4), we first find the optimal solution to the LP relaxation: $x_1 = \frac{13}{7}, x_2 = 0$. Rounding this solution yields the solution $x_1 = 2, x_2 = 0$ as a possible



optimal solution to (4). But $x_1 = 2, x_2 = 0$ is infeasible for (4), so it cannot possibly be the optimal solution to (4). Even if we round x_1 downward (yielding the candidate solution $x_1 = 1, x_2 = 0$), we do not obtain the optimal solution ($x_1 = 0, x_2 = 3$ is the optimal solution).

Heuristic optimum may not be truly optimal, but it does satisfy all constraints and possessing good objective values. Where time permits no deeper analysis, this rounded relaxation solution might well suffice.

Principle 2.6 The objective function value of any (integer) feasible solution to a maximizing discrete optimization problem provides a lower bound on the integer optimal value. Any (integer) feasible solution to a minimizing discrete optimization problem provides an upper bound.

Therefore, for a general maximize ILP we have:

$$z_{int} \leq z_{int}^{*} \leq z_{LP}^{*}.$$

for a general minimize ILP we have:

$$z_{LP}^{*} \leq z_{int}^{*} \leq z_{int}$$

Exercise
Find an upper and a lower bound for the problem

$$\begin{aligned} \max \quad & 7x_1 + 12x_2 + 5x_3 + 14x_4 \\ \text{s.t.} \quad & 10x_1 + 20x_2 + 8x_3 + 22x_4 \leq 35, \\ & x_1, x_2, x_3, x_4 \in \{0, 1\} \end{aligned}$$

Sharp LP Relaxations and Convex Hull

Suppose that we are concerned with the LP relaxation of IP/MIP. The difficulty is that integer feasible solutions to LP problems are more likely to lie in the interior of the feasible region than on its boundary.

Graphically, what lies between integer solutions and constraint lines are points introduced by the relaxation.

Definition 2.7 If a LP relaxation is so good that all its feasible region vertices are integer, the solution of LP will be integer and would be that of the IP as well. In this case we call the resulting LP relaxation Sharp/Ideal Relaxation.

The following example illustrates different representations of a feasible (integer) set by linear inequality and integrality restrictions.

Example:
Consider the constraint $4x_1 + 6x_2 \leq 9$, with $x_1, x_2 \geq 0$ and integer. Suppose we are maximizing $f = x_1 + x_2$ subject to this constraint, by solving its LP relaxation (graphically) we obtain

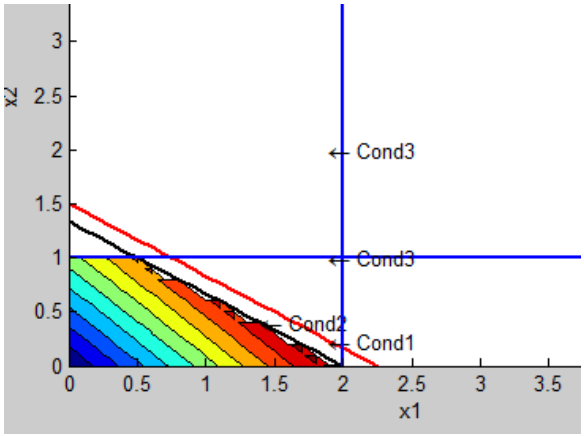
$$x^*_{LP} = (\frac{9}{4}, 0)^T, f(x^*_{LP}) = \frac{9}{4}.$$

However, if the feasible region were described by constraints

$$2x_1 + 3x_2 \leq 4$$
$$x_1 \leq 2$$
$$x_2 \leq 1$$
$$x_1,\ x_2 \geq 0 \text{ and integer}$$

The optimal solution to the LP relaxation of the refined model now becomes: Which solves the original IP.

$$\tilde{x}^*_{LP} = (2,0)^T, f(\tilde{x}^*_{LP}) = 2,$$



(Notice that what we have done is generating some ”additional” constraints that are directly implied by the original constraint together with integrality restrictions.)

Surprisingly, in IP there is often advantage in increasing rather than decreasing the number of integer variables and/or constraints in a model. Another typical example is the **uncapacitated Facility Location problem**

$$\min \quad \sum_{j \in N} c_j\, x_j + \sum_{i \in I} \sum_{j \in N} h_{ij} y_{ij}$$
$$s.t. \quad \sum_{j \in N} y_{ij} = 1, \ \ \forall i \in I \quad \text{(demand conditions)}$$
$$y_{ij} \leq x_j\,, \ \ \forall i \in I, \ \ \forall j \in N \ \ (service\ conditions)$$
$$x \in \{0,1\}^n$$
$$y \in [0,1]^{m \times n}$$

It is fairly straightforward to see that service conditions in above model implies

$$\sum_{i \in I} y_{ij} \leq |I| x_j, \quad \forall j \in N$$

However, practice shows that the original model largely decreases the difficulty of solution process although it has far more constraints than the modified one. Intuitively more constraints give a sharper definition of the feasible space.
Note: يوجد خطا في تفسير النظام اعلاه في الفديو حيث كان المقصود به ان النظام الاصلي اسهل بالحل على الرغم من كثرة القيود وعندما تغير القيد الثاني ليعطي عدد قيود اقل اصبح النظام اكثر صعوبة.

Remarks
• Actually the unique **sharp relaxation** for above example is given by:

$$x_1 + 2x_2 \leq 2$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

• Indeed, the sharp relaxation is the best one can achieve in terms of modelling. Nevertheless, in practice it is not easy to generate.
• Theoretical proof of the existence of a set of linear inequalities for the sharp description will be covered later in **Polyhedral Theory**.

2.2 Pre-processing and Model Improvement

Pre-processing is the phase between model formulation and model solution. It helps to successfully solve models and to do it in a shorter time.

Before attempting to solve a problem, we want to have it formulated the best we can. Although reducing dimension of the problem may not be introducing an easier model to solve, sometimes we still need such pre-processing steps, e.g. make the model more compact or more extensive as some solvers may accept problems with only restricted number of variables and/or constraints.

Reduce Dimensionality - remove redundant <u>variables</u>

The number of variables can be reduced by:
• Fixing some variables to constant values that our knowledge of the problem may suggest or as a temporary measure in the solution process;

• Exploiting equations (if there are any) and use them to write some variables in terms of others. استخدام بعض المعادلات لكتابة متغيرات بدلالة متغيرات اخرى
However, this may lead to additional number of constraints.

Example:
Consider the following Facility Location problem

$$\begin{aligned} \min \quad & 16x_1 + 20x_2 + 2y_{11} + 3y_{12} + 2y_{21} + y_{22} \\ s.t. \quad & y_{11} + y_{21} = 1 \\ & y_{12} + y_{22} = 1 \\ & y_{11}, y_{12} \leq x_1 \\ & y_{21}, y_{22} \leq x_2 \\ & x_1, x_2 \in \{0,1\} \\ & y_{11}, y_{12}, y_{21}, y_{22} \in [0,1] \end{aligned}$$

Notice that there are two equality constraints, which allows us to rewrite y_{21}, y_{22} as

$$y_{21} = 1 - y_{11}$$
$$y_{22} = 1 - y_{12}$$

Therefore, we can eliminate y_{21}, y_{22} and the system becomes a minimization problem with four variables:

$$\begin{aligned} \min \quad & 16x_1 + 20x_2 + 2y_{11} + 3y_{12} + 2(1 - y_{11}) + (1 - y_{12}) \\ s.t. \quad & y_{11}, y_{12} \leq x_1 \\ & x_1, x_2 \in \{0,1\} \\ & y_{11}, y_{12} \in [0,1] \end{aligned}$$

However, by doing this what we have deleted together with the elimination of y_{21}, y_{22} are $y_{21}, y_{22} \leq x_2$ and their boundary conditions: y_{21}, $y_{22} \in [0,1]$.

$$y_{21} = 1 - y_{11} \leq x_2$$
$$y_{22} = 1 - y_{12} \leq x_2$$
$$0 \leq y_{21} = 1 - y_{11} \leq 1$$
$$0 \leq y_{22} = 1 - y_{12} \leq 1.$$

Therefore, the entire simplified model is given by:

$$\begin{aligned} \min \quad & 16x_1 + 20x_2 + 2y_{11} + 3y_{12} + 2(1 - y_{11}) + (1 - y_{12}) \\ s.t. \quad & y_{11}, y_{12} \leq x_1 \\ & y_{11}, y_{12} \geq 1 - x_2 \\ & x_1, x_2 \in \{0,1\} \\ & y_{11}, y_{12} \in [0,1] \end{aligned}$$

Reducing the dimension of model can sometimes be seen as a way forward to ease what is commonly known as the **curse of dimensionality**.